

D-branes in non-critical superstrings and duality in $\mathcal{N} = 1$ gauge theories with flavor

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ABSTRACT: We study D-branes in the superstring background $\mathbb{R}^{3,1} \times SL(2, \mathbb{R})_{k=1}/U(1)$ which are extended in the cigar direction. Some of these branes are new. The branes realize flavor in the four dimensional $\mathcal{N} = 1$ gauge theories on the D-branes localized at the tip of the cigar. We study the analytic properties of the boundary conformal field theories on these branes with respect to their defining parameter and find non-trivial monodromies in this parameter. Through this approach, we gain a better understanding of the brane set-ups in ten dimensions involving wrapped NS5-branes. As one application, using the boundary conformal field theory description of the electric and magnetic D-branes, we can understand electric-magnetic (Seiberg) duality in $\mathcal{N} = 1$ SQCD microscopically in a string theoretic context.

KEYWORDS: Conformal Field Models in String Theory, Brane Dynamics in Gauge Theories, Duality in Gauge Field Theories.

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1. Introduction and summary

Non-critical superstring theory [1] is a part of the moduli space of string theory compactifications with special properties. The dimension of space-time can be lowered, the background contains fewer (super)symmetries, and there are fewer fields in the perturbative string spectrum. The search for (linear dilaton) holography in lower dimensions, or the exploration of the full moduli space of string theory compactifications are sufficient motivation for the study of these backgrounds. In this paper, we concentrate on their ability to allow for the economical study of $\mathcal{N} = 1$ gauge theories with flavor (see e.g. [2–5] and [6, 7]).

Many properties of gauge theories have gained intuitive interpretations via their embedding into string theory using D-branes. One such field where progress was made in understanding Seiberg duality in $\mathcal{N} = 1$ gauge theories is in terms of brane set-ups [8, 9]. In particular the matching of the moduli spaces and the chiral rings was achieved in terms of the embeddings of NS5-branes and D-branes in flat space-times.

In this paper, we wish to develop the understanding of $\mathcal{N} = 1$ gauge theories via their embedding into string theory further. In particular, the non-critical superstring theory set-up automatically takes into account the backreaction of the NS5-branes (of mass $1/g_s^2$) on the closed string background, in the (doubly scaled) near-horizon limit [10]. This allows us to clearly separate effects due to the closed string NS5-brane solitons, and due to the D-brane defects (of mass $1/g_s$). Indeed, once the closed string backreaction due to the NS5-branes is taken care off, we can study the D-branes in their presence using the boundary states that code the boundary conformal field theory that the open strings give rise to.

We will argue in this paper that this provides a microscopic view on D-brane set-ups and on Seiberg duality [11]. In particular, we will see how the microscopy allows to clearly see the appearance and importance of the meson in the magnetic dual, the phenomenon of Higgsing, and the appearance of dangerously irrelevant operators in the field theory. This involves a nice but detailed interplay between the boundary SCFT's on the worldsheet, the physics of the gauge theory and the closed string physics in these highly curved backgrounds. In the rest of this section, we summarize the flow of the main ideas in the paper.

We study the type IIB d dimensional closed non-critical superstring [1] background $\mathbb{R}^{d-1,1} \times SL(2, \mathbb{R})/U(1)$. The factor $SL(2)_k/U(1)$ is a Kazama-Suzuki coset superconformal field theory [12], which has a mirror description as the $\mathcal{N} = 2$ Liouville theory. The supercoset has central charge $c = 3 + \frac{6}{k} = 3(1 + Q^2)$, where Q is the slope of the linear dilaton theory that the conformal field theory asymptotes to. We shall focus on the level $k = 1$ which comes with a corresponding four-dimensional ($d = 4$) flat space factor.

We shall study D-branes which fill the flat space $\mathbb{R}^{3,1}$. The low energy theories on the worldvolume of these D-branes are the four-dimensional minimally supersymmetric gauge theories.¹ The field content and interactions of the gauge theory depend on the profile of

¹The branes which are Dirichlet in some of the flat directions are also interesting, they give rise to non-perturbative effects like domain walls and instantons [7] in the gauge theories.

the brane in the cigar. The branes on the cigar are of two types, localized and extended². The localized branes are based on the identity representation and live at the tip of the cigar. They realize pure super Yang-Mills as their world-volume theories. These were analyzed in [6] and [7] independently (see also [17]). Here, we shall study further the extended branes which realize flavor in $\mathcal{N} = 1$ gauge theories [6, 18].

The extended branes obey to a good approximation Neumann boundary conditions in the weak-coupling region. As we go towards the strong coupling region, a worldvolume potential develops and they dissolve away. These branes are defined semi-classically by a complex parameter μ_B whose absolute value indicates how far they extend into the region near the tip. Quantum mechanically, they are better described by the parameters (J, M) which are the labels of the $SL(2)$ representations from which they descend.

For the continuous representation $(J = \frac{1}{2} - iP, M = \frac{1}{2})$, the branes introduce quarks and anti-quarks into the gauge theory [6]. The mass of the quarks is set by the parameter P^2 , and vanishes at $P^2 = 0$. For other values of (J, M) , the corresponding boundary states set down in [17, 19, 20] were studied in detail in [21]. One aspect which was not very clear about these branes was whether they have a well-defined unitary self-overlap.³ We find here that one can indeed understand these branes systematically following the ideas of [15] and there do exist branes with a unitary spectrum for values of J in the range $0 \leq J \leq \frac{1}{2}$.

These branes have a semiclassical interpretation of turning on a worldvolume two form flux in the region near the tip of the cigar. At $J = 0$, this value of the flux reaches a critical value and localizes at the tip, corresponding to forming a localized brane. In the quantum theory, this is seen in a relation between the boundary states which expresses the localized identity boundary state as a difference between the $J = M = 0$ and the $J = M = \frac{1}{2}$ boundary state. This phenomenon is similar to that in Liouville theory [22]; the $\mathcal{N} = 2$ Liouville theory interprets this relation as the localized brane being “a brane inside a brane” [23] i.e. built of gauge fields living on the extended brane.

The quantum description of the branes as boundary states contains much more information. The $J = M = 0$ brane introduces flavors in the fundamental and anti-fundamental representation, along with a gauge field into the gauge theory on the localized branes. Its self overlap contains, among other modes a localized mode with the quantum numbers of a meson!

We thus develop a picture of the Higgs mechanism in this theory as the following: the $J = \frac{1}{2}$ flavor brane combines with the color brane to give the $J = 0$ flavor brane. The $J = 0$ flavor brane thought of as a single object preserves only one combination of the two $U(1)$'s rotating the two branes independently. It sits at the origin of the Higgs branch of the theory and realizes in its spectrum the meson as a Nambu-Goldstone boson of the broken symmetry.

From what we said above about the relation between the three branes (all of positive tension), it might seem like the $J = \frac{1}{2}$ brane and the identity brane should be thought of (outside the Higgs branch) as the elementary branes. However, this turns out to be so

²These are similar to the ZZ [13] and FZZT [14, 15] branes in Liouville theory; in fact the relation between them is more than an analogy [16].

³It was noticed [17] that for real values of $J \in (\frac{1}{4}, \frac{1}{2}]$, the self-overlaps are well-defined.

because of a particular choice in orientation of the branes with respect to the closed string background⁴. This change of orientation is implemented in the theory by changing the phase of the bulk parameter μ which multiplies the $\mathcal{N} = 2$ Liouville potential.

The operation $\mu \rightarrow -\mu$ affects all the states in the theory including the branes, and the net effect is that the new color brane and the $J = 0$ brane become elementary and the $J = \frac{1}{2}$ brane can be expressed as a sum of these two. Following this phenomenon onto the low energy theory realized on the branes leads us to the electric-magnetic duality of Seiberg [11]. This type of rearrangement of the basis of boundary states leading to Seiberg duality was observed at the level of the charge lattice in [24–27] for quiver gauge theories. With an explicit worldsheet description of our theories and their branes at hand, we can lift these statements to the exact boundary states which includes the coupling to all the closed string excitations of the theory.

The boundary state description has one more piece of easily computable, useful information. It tells us how the closed string background backreacts to the presence of the source. It has been argued (see e.g. [28]) that the profile of the backreaction encodes the running of the coupling constant in the field theory. Such considerations in this case tell us that the gauge theory in question has indeed the spectrum of $\mathcal{N} = 1$ SQCD, but also suggest that there is an interaction quartic in the quark superfields.⁵ This theory can be thought of as a softly broken $\mathcal{N} = 2$ theory which is sensible from the viewpoint of 5-brane embeddings in string theory as we shall explain.

The layout of the rest of the paper is as follows: In section 2, we summarize the construction of the various boundary states on the cigar, and then present the addition formulas. In section 3, we develop a semiclassical understanding of the various branes and their addition relations, and then present the transformation properties under the change in phase of μ referred to above. In section 4, we present the self-overlaps of the various branes, paying attention to the new ones — we show the appearance of the localized modes, and explain their role in the gauge theory. We then address the question of mutual supersymmetry and present a list of branes which preserve the same set of supersymmetries. In section 5, we explain the relation between our theories and the more familiar brane set-ups in ten dimensions and point out where the exact treatment is not just pleasing but necessary.

Having thus assembled all the ingredients and intuitions, we put them all together in section 6 and read off as a consequence the duality in the low energy theory. Then, in section 7, we begin to develop an understanding of the processes as an RG flow in the boundary CFT. In section 8, we probe in more detail the interactions in the gauge theory under consideration. We compute a first order backreaction of the branes onto the closed string background and show that the anomalies of a classical $U(1)$ global symmetry of the theory is encoded in the backreaction onto the RR axion. From the backreaction onto the NSNS sector, we argue that the gauge theory under consideration is really $\mathcal{N} = 1$ SQCD

⁴The relative orientation arises while specifying the sine-Liouville potential to define the worldsheet theories and is a well-defined notion because of the lack of further transverse directions, as is familiar from set-ups with anomalous creation of branes.

⁵This is also consistent with the lack of a chiral flavor symmetry in the theory.

with a quartic coupling of the quarks. In section 9 , we make some final remarks and point out open issues.

2. The branes on the cigar

The boundary states on the $\mathcal{N} = 2$ supercoset $SL(2, \mathbb{R})/U(1)$ can be classified as A type or B type according to the gluing of the left-moving $\mathcal{N} = 2$ algebra to its right-moving counterpart. When tensored with space filling boundary states in the flat space $\mathbb{R}^{3,1}$, the B-branes in the cigar are supersymmetric in the type IIB closed superstring theory. Since the $U(1)$ R -current of the $\mathcal{N} = 2$ superconformal algebra on the worldsheet contains a term that translates the chiral boson corresponding to the angular direction θ of the cigar, the B-type boundary condition always leads to Neumann boundary conditions in the angular direction θ , i.e. the B-branes conserve the momentum around the cigar. If a brane extends to the asymptotic weak coupling region where a semi-classical notion is valid, we can say that the supersymmetric branes wrap the circle of the cigar. We will sometimes use the picture of the T-dual type IIA superstring theory on the $\mathcal{N} = 2$ Liouville theory with a momentum sine-Liouville condensate in which the supersymmetric A-branes conserve the winding around the cylinder.

A large class of consistent boundary states has been presented in a clear manner in [21] , summarizing and extending earlier work [17, 19, 20]. The consistency conditions on the boundary states include the requirement that the spectrum of boundary operators be sensible (e.g. has a positive density of states), as well as a factorization equation which leads to a so-called shift-equation, analyzed in [29, 21]. A complete check of the consistency requirements on these non-rational boundary conformal field theories remains to be performed.

It is crucial [17] that the set of B-branes contains an identity brane that by definition has only the identity representation in its open string spectrum. Its overlap with other B-branes contains (as a consequence of a generalized Verlinde formula [30]) only the single representation characterizing those branes. Thus, we can review first the relevant characters of the representations of the $\mathcal{N} = 2$ superconformal algebra, then recall the associated D-branes.

The relevant representations are actually representations of the $\mathcal{N} = 2$ superconformal algebra extended by a spectral flow operation, leading to extended characters i.e. characters summed over spectral flow orbits [17] . It is convenient to label these characters in terms of quantum numbers associated to a parent supersymmetric SL_2 theory. The characters are then labeled by the numbers (J, M) , parameterizing their parent SL_2 Casimir and compact $U(1)$ charge. Equivalently they can be labeled by their $\mathcal{N} = 2$ conformal dimension and R-charge (h, Q) , where these are given in terms of the quantum numbers (J, M) by the formulas $h = -\frac{J(J-1)}{k} + \frac{M^2}{k}$ and $Q = \frac{2M}{k}$ in the NS sector. Another convenient parameterization, natural for continuous representations of SL_2 , is given by the formula $J = \frac{1}{2} - iP$, where P is a momentum variable. For a detailed discussion of these issues see [31, 19].

In order not to clutter the formulas, in the following we immediately restrict to our case of interest, namely the level k of the coset equals one. For $k = 1$, the relevant extended characters in the NS sector are labeled by a charge $Q \in \{0, \pm 1\}$. We distinguish representations according to whether they descend from continuous or discrete representations, or the identity representation in the parent SL_2 theory:

$$\begin{aligned}
 \textbf{Continuous} : \quad & J = \frac{1}{2} - iP, P \in \mathbb{R}^+, M \in \{0, \frac{1}{2}\} \\
 Ch_{\text{cont}}^{\text{NS}}(h, Q; \tau, z) = & q^{h - \frac{Q^2+1}{4}} \sum_{m \in \mathbb{Z}} q^{(m+\frac{Q}{2})^2} y^{2m+Q} \frac{\vartheta_{00}(\tau, z)}{\eta^3(\tau)} \\
 \textbf{Discrete} : \quad & J = |M| \\
 Ch_{\text{disc}}^{\text{NS}}(Q; \tau, z) = & q^{-\frac{1}{4}} \sum_{m \in \mathbb{Z}} \frac{q^{m^2+|Q|m+\frac{|Q|}{2}} y^{\text{sgn}(Q)(2m+|Q|)}}{(1+y^{\text{sgn}(Q)}q^{m+\frac{1}{2}})} \frac{\vartheta_{00}(\tau, z)}{\eta^3(\tau)} \\
 \textbf{Identity} : \quad & J = M = 0 \\
 Ch_{\text{Id}}^{\text{NS}}(\tau, z) = & q^{-\frac{1}{4}} \sum_{m \in \mathbb{Z}} \frac{(1-q)q^{m^2+m-\frac{1}{2}} y^{2m+1}}{(1+yq^{m+\frac{1}{2}})(1+yq^{m-\frac{1}{2}})} \frac{\vartheta_{00}(\tau, z)}{\eta^3(\tau)}. \tag{2.1}
 \end{aligned}$$

Note that the continuous characters depend on $|Q|$ only. The characters form a representation of the modular group [17]. The characters in the other sectors $\tilde{N}S, R, \tilde{R}$ can be found by $\mathcal{N} = 2$ spectral flow (see e.g. [17, 19]).

Corresponding to each of these characters, there is a consistent boundary state [21]. These boundary states are constructed in the following manner. We first define the Ishibashi states:

$$\langle\langle p, Q | e^{-\pi T H^{\text{cl}}} e^{i\pi z(J+\tilde{J})} | p, Q' \rangle\rangle = 2\pi (\delta(p-p') + \delta(p+p')) \delta_{\mathbb{Z}_2}(Q, Q') Ch^{\text{NS}}(p, Q, iT, z). \tag{2.2}$$

Then the identity brane corresponds to the one-point functions⁶:

$$\begin{aligned}
 |B; Id\rangle &= \int_{-\infty}^{\infty} \frac{dp'}{2\pi} \sum_{Q' \in \mathbb{Z}_2} \Psi_{\text{Id}}(p', Q') |p', Q'\rangle \\
 \Psi_{\text{Id}}(p', Q') &= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \nu^{ip'} \frac{\Gamma(\frac{1}{2} + \frac{Q'}{2} + ip') \Gamma(\frac{1}{2} - \frac{Q'}{2} + ip')}{\Gamma(i2p') \Gamma(1 + i2p')}. \tag{2.3}
 \end{aligned}$$

It can be checked that it has only the identity character in its self-overlap. The brane associated to the continuous representation has a Cardy state:

$$\begin{aligned}
 |B; \text{cont}, P, Q\rangle &= \int_{-\infty}^{\infty} \frac{dp'}{2\pi} \sum_{Q' \in \mathbb{Z}_2} \Psi_{\text{cont}}(p', Q') |p', Q'\rangle \\
 \Psi_{\text{cont}}(p', Q') &= (2\pi)^{\frac{1}{2}} \nu^{ip'} \cos(4\pi pp') \frac{\Gamma(1 - i2p') \Gamma(-i2p')}{\Gamma(\frac{1}{2} - ip' + \frac{Q'}{2}) \Gamma(\frac{1}{2} - ip' - \frac{Q'}{2})} e^{\pi i Q Q'}. \tag{2.4}
 \end{aligned}$$

⁶We have included here the dependence on the bulk interaction coefficient ν which we assume to be real for now. We discuss the physics of its phase later.

The brane carries only the continuous representation in its overlap with the identity brane. Finally, the brane carrying only the discrete representation in its overlap with the identity brane can be described by the formulas:

$$\begin{aligned}
 |B; disc, Q\rangle &= \int_{-\infty}^{\infty} \frac{dp'}{2\pi} \sum_{Q' \in \mathbb{Z}_2} \Psi_{disc}(p', Q') |p', Q'\rangle \\
 \Psi_{disc}(p', Q') &= i(8\pi)^{-\frac{1}{2}} \nu^{ip'} e^{i\pi QQ'} \Gamma(1 - 2ip') \Gamma(-2ip') \times \\
 &\left(e^{i\pi(\frac{Q'}{2} - \frac{1}{2} + ip')} e^{-i4\pi p'p} \frac{\Gamma(\frac{1}{2} + ip' + \frac{Q'}{2})}{\Gamma(\frac{1}{2} - ip' + \frac{Q'}{2})} - e^{i\pi(\frac{Q'}{2} + \frac{1}{2} - ip')} e^{i4\pi p'p} \frac{\Gamma(\frac{1}{2} + ip' - \frac{Q'}{2})}{\Gamma(\frac{1}{2} - ip' - \frac{Q'}{2})} \right) \quad (2.5)
 \end{aligned}$$

This is the anti-chiral brane in [21]. There is another discrete brane corresponding to the chiral brane in [21]. The one-point functions in the other sectors $\tilde{N}S, R, \tilde{R}$ can be obtained by spectral flow. It was checked in [21], and further in [32] that the branes listed above obey a bulk-boundary factorization constraint (i.e. the shift equation).

There are restrictions on the values of the parameters (J, M) labeling the Cardy states. The open string spectrum⁷ allows only for unitary representations to appear in the cigar brane overlap. Moreover, we need to demand mutual consistency of the boundary states as well as with the bulk spectrum of the theory. Our strategy will be to consider the branes to be analytic functions of the parameters (J, M) in some of the calculations that follow, and take care to impose all restrictions that follow from consistency and unitarity.

2.1 Addition relations obeyed by branes

For the level $k = 1$, the extended characters thought of as analytic functions of (J, M) obey two identities [17]:

Character addition formulas

$$\begin{aligned}
 Ch_{cont}(h = \frac{1}{2}, |Q| = 1; \tau, z) &= Ch_{disc}(Q = 1; \tau, z) + Ch_{disc}(Q = -1; \tau, z) \\
 Ch_{cont}(h = 0, Q = 0; \tau, z) &= Ch_{cont}(h = \frac{1}{2}, |Q| = 1; \tau, z) + Ch_{Id}(\tau, z). \quad (2.6)
 \end{aligned}$$

What is even more powerful is that the corresponding branes obey related identities which can be checked by using the explicit one-point functions in formulas (2.3), (2.4), and (2.5). Using these we obtain the important addition relations between the branes:

Brane addition formulas

$$\begin{aligned}
 |B; cont, h = \frac{1}{2}, |Q| = 1\rangle &= |B; disc, Q = 1\rangle + |B; disc, Q = -1\rangle \\
 |B; cont, h = |Q| = 0\rangle &= |B; cont, h = \frac{1}{2}, |Q| = 1\rangle + |B; Id\rangle. \quad (2.7)
 \end{aligned}$$

In the next section, we shall develop a semiclassical spacetime understanding ($g_s \rightarrow 0$) of the various branes and in particular the above two equations in (2.7)⁸. We remark

⁷In this particular set-up a light-cone gauge choice in some extra flat directions is possible.

⁸By the modular bootstrap reasoning, the brane addition formula (2.7) implies the character addition formula (2.6).

already that the second addition relation is similar to an equation for boundary states in bosonic $c = 1$ theory⁹[33, 22, 34]. We will understand its implications to gauge theory physics in the following sections. In the rest of this section, we shall summarize the open string sigma-model ($\alpha' \rightarrow 0$) understanding of the branes.

2.2 The branes as defined by the Boundary cosmological constant

Let us recall that the bulk theory is defined by a complex parameter, the coefficient $\mu, \bar{\mu}$ of the $\mathcal{N} = 2$ Liouville coupling $\mathcal{L}^{\text{SL}} = \mu \psi \tilde{\psi} e^{-\frac{1}{Q}(\rho + \tilde{\rho} + i(\theta - \tilde{\theta}))} + c.c$ where we have used the asymptotic variables on the cigar. There is also the related parameter $\tilde{\mu}$, the coefficient of the cigar interaction in terms of which the correlators can be defined. For $\mu = \bar{\mu}$, the equation relating the coupling constants is [21] :

$$(g_s^{\text{tip}})^{-2} = \mu^{2/k} = \tilde{\mu} \frac{\Gamma(\frac{1}{k})}{\Gamma(1 - \frac{1}{k})} \equiv \nu. \tag{2.8}$$

The case of level $k = 1$ needs a renormalization [7] similar to the Liouville theory at $b = 1$ [35], as can be seen from the bulk tachyon reflection amplitude. We have, with $k = 1 + \epsilon$,

$$(g_{s,\text{ren}}^{\text{tip}})^{-2} = \mu_{\text{ren}}^2 = \nu_{\text{ren}}; \quad \mu_{\text{ren}} = \mu \epsilon \tag{2.9}$$

On the boundary, the coupling constants are the cosmological constants $\mu_B, \bar{\mu}_B$ and the non-chiral coupling $\tilde{\mu}_B$, related to the brane labels (J, M) by the equations [21] :

$$\begin{aligned} \mu_B &= \left(\frac{2k\mu}{\pi}\right)^{1/2} \sin(\pi(J - M)) \\ \bar{\mu}_B &= \left(\frac{2k\mu}{\pi}\right)^{1/2} \sin(\pi(J + M)) \\ \nu_B &\equiv \tilde{\mu}_B \frac{\Gamma(\frac{1}{k})}{\Gamma(1 - \frac{1}{k})} = -\frac{\mu \Gamma(\frac{1}{k})}{2\pi} \cos\left(\frac{\pi}{k}(2J + 1)\right) \end{aligned} \tag{2.10}$$

For the level $k = 1$ then, the renormalized cosmological constants (which enter the open string amplitudes) are defined by the formulas:

$$\begin{aligned} \mu_B \epsilon^{1/2} &\equiv \mu_{B,\text{ren}} = \left(\frac{2\mu_{\text{ren}}}{\pi}\right)^{1/2} \sin(\pi(J - M)) \\ \bar{\mu}_B \epsilon^{1/2} &\equiv \bar{\mu}_{B,\text{ren}} = \left(\frac{2\mu_{\text{ren}}}{\pi}\right)^{1/2} \sin(\pi(J + M)) \\ \nu_B \epsilon &\equiv \nu_{B,\text{ren}} = -\frac{\mu_{\text{ren}}}{2\pi} \cos(\pi(2J + 1)) = \frac{\mu_{\text{ren}}}{2\pi} \cos(2\pi J). \end{aligned} \tag{2.11}$$

3. Semi-classics and covariance

In this section, we discuss some of the semi-classical properties of the D-branes in the cigar, and the covariance properties of the boundary states under a Z_2 operation.

⁹Other precise relations between the branes of the two theories were written in [16].

3.1 Semi-classical description of the branes

We observe that the D1-branes of sine-Liouville theory with momentum condensate (see e.g. [6]) reach asymptotic infinity from two angular directions, with an angular parameter θ_0 that takes values in a full circle of length 2π . When we concentrate on the brane that stays fixed in the angular direction as it goes to the strongly coupled region ($r = 0$) however, we note that the NSNS sector one-point function is invariant under the operation $\theta_0 \rightarrow \theta_0 + \pi$. This is directly associated to the fact that the D1-branes have two legs, and that it allows for half-integer winding open strings. In the full supersymmetric theory, we would moreover pick up a sign in the RR term in the boundary state under this operation, due to the difference in orientation of the resulting rotated D1-brane. In contrast, the localized D1-branes of [32] (which we may think of as difference of (anti-)chiral branes in the nomenclature of [21]) are only invariant under the full 2π rotation, showing that they have, in this sense, only a single leg (which in the case of these D1-branes is localized near the more strongly coupled region).

The picture we sketched above is T-dual to the D2-branes on the cigar whose physics we can now more easily picture. For D2-branes, the differences discussed above are reflected in having either a single-sheeted D2-brane, or a double-sheeted D2-brane [6]. The associated Wilson line on the D2-brane is either 2π or π valued. Our discrete brane (2.5) which can be identified with the D2-brane of [36] and the anti-chiral brane of [21] is single sheeted. In contrast the continuous brane consist of two oppositely oriented sheets (represented by the sum of a chiral and an anti-chiral brane). Thus, we have given a geometrical picture for the first addition relation in (2.6).

The picture we developed also explains why the single-sheeted discrete D2-branes give rise in the full theory to a RR-tadpole that cannot be absorbed in the background. It leads to an inconsistent bulk theory (for space-filling branes in the flat directions). By open-closed string duality, should we add such a brane to the theory, we would discover anomalous chiral matter in the massless open string sector. In contrast, the two-sheeted continuous brane cleverly cancels the potential RR-tadpole by having two sheets of opposite orientation (nevertheless preserving supersymmetry). See also [6].

The second addition relation shows that the difference between the two particular two-sheeted space-filling branes under study is precisely the brane localized at the tip. The brane $J = \frac{1}{2} + iP$ extends from infinity to a certain distance determined by P^2 from the tip where it dissolves, as can be seen from the one-point functions [6]. As $P = 0$, it covers the whole cigar. On adding a localized brane at the tip, we get formally the $J = 0$ brane.

We can understand this “addition” by realizing that the localized brane sources a two-form flux on the extended brane. The $J = 0$ brane admits a deformation where the cigar is still covered, but the distribution of the two form field changes. On turning on this deformation, the B-field spreads out from the tip and localizes in a ring at a small distance from the tip [37]. Semi-classically, we can identify this parameter to be $J \in [0, \frac{1}{2}]$. The $J = 0$ brane then is simply a point in this moduli space where the dissolved brane becomes point-like at the tip.

3.2 Dependence of the branes on the bulk interaction parameter μ , and the action of $\mu \rightarrow -\mu$.

In this subsection, we want to study how the closed string modes and the D-branes change under a change of phase of the bulk interaction parameter μ , and in particular under $\mu \rightarrow -\mu$. The action $\mu \rightarrow -\mu$ is not a symmetry of the $\mathcal{N} = 2$ theory as can be seen from the closed string interaction written in section 2.2. However, the covariance of the theory tells us how the objects in the theory defined by μ are related to the objects in the theory with $-\mu$. From the form of the worldsheet action, we also see that the operation $\mu \rightarrow -\mu$ is equivalent to the action $(-)^w : \theta \rightarrow \theta + \pi$ and $\tilde{\theta} \rightarrow \tilde{\theta} - \pi$ on the left-moving and right-moving angular coordinate in the cigar theory. In the T-dual sine-Liouville picture, this is a rotation of the cylinder by π .

In the perturbative sector of the theory, closed string modes in the sector with odd values of asymptotic winding pick up a sign, and those with even values of winding do not. The tachyon state of winding one which has a condensate in the theory picks up a sign under the operation.

Now we ask what is the action on the boundary states. Firstly, there is an implicit dependence through the coupling to the closed string modes. We can use the same Ishibashi basis (2.2) as before and keep track of the sign dependence as a phase in the one-point function. The identity brane depends only on the closed string parameter μ , and no other intrinsic parameter, and so the same should be true about its corresponding one point function. We can write the μ dependence as:

$$\Psi_{\text{Id}}(p', Q'; \mu) = \mu^{ip' - \frac{Q'}{2}} \frac{1}{\mu^{ip' + \frac{Q'}{2}}} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2} + \frac{Q'}{2} + ip') \Gamma(\frac{1}{2} - \frac{Q'}{2} + ip')}{\Gamma(i2p') \Gamma(1 + i2p')} \quad (3.1)$$

which is consistent with the bulk reflection amplitude [17]. Using the one-point functions (2.4), (2.5), we can also write a similar equation for the μ dependence of the extended branes.

In addition, the action of the rotation $(-1)^w$ could induce explicit changes of sign due to the full boundary state having a well-defined charge under the above mentioned symmetry $\mu \rightarrow -\mu$ accompanied by $(-1)^w$. To test this, we look at the coupling of the on-shell tachyon winding mode with the localized and extended branes. From (2.3), (2.4), we see that the former couples (with an infinite coefficient (see [7])) and the extended brane $|B; J = M = \frac{1}{2}\rangle$ has vanishing coupling. We deduce the following transformations under $\mu \rightarrow -\mu$:

$$\begin{aligned} |B; Id; \mu\rangle_{\text{NS}} &\rightarrow -|B; Id; -\mu\rangle_{\text{NS}} \\ |B; J = M = \frac{1}{2}; \mu\rangle_{\text{NS}} &\rightarrow |B; J = M = \frac{1}{2}; -\mu\rangle_{\text{NS}} \end{aligned} \quad (3.2)$$

Note that the brane $|B; J = M = 0\rangle$ which is a linear combination of the above two does not have a fixed transformation property under the operation.

It is important to note that the tension of the new localized brane $(-|B; Id; -\mu\rangle_{\text{NS}})$ is positive — this can be seen by the computation of [7] where the tension was worked out to

be proportional to $\nu_{\text{bulk}}^{\frac{1}{2}} = \mu_{\text{bulk}}$. A change in the overall sign of the boundary state and a change in sign of μ together give a factor of unity. At an intuitive level, we have that the tension of the brane localized at the tip is proportional to $g_s^{\text{tip}^{-1}} = \mu_{\text{bulk}}$ (2.9). A change in its sign forces a change in sign of the positive tension boundary state.

Let us now discuss the RR sector of the boundary states. Under the rotation of the cylinder with a sine-Liouville potential by π , the full state $|B; J = M = \frac{1}{2}\rangle$ changes orientation with respect to the potential. This orientation does not make a difference for the NS sector of the boundary state, but it implies that the RR sector state changes sign with respect to the RR sector closed string fields on the cylinder (e.g. RR one-form in type IIA).¹⁰ We will see later that this is consistent with the NS5-brane setup in ten dimensions. We write then the final equations describing the transformations of the branes under $\mu \rightarrow -\mu$:

$$\begin{aligned} |B; Id; \mu\rangle &\rightarrow -|B; Id; -\mu\rangle \\ \hline |B; J = M = \frac{1}{2}; \mu\rangle &\rightarrow |B; J = M = \frac{1}{2}; -\mu\rangle. \end{aligned} \tag{3.3}$$

4. The spectrum of open strings ending on various branes

In this section, we shall address the issue of the self-overlaps of the continuous branes, the overlaps between the localized and continuous branes, and the mutual supersymmetries preserved by the various branes. At the end of the section, we will present a summary of branes which are mutually supersymmetric and have a unitary spectrum in their overlaps.

4.1 The self overlap of the extended branes

As mentioned earlier, we can formally define branes based on the continuous and discrete characters with arbitrary values¹¹ of (J, M) . A necessary condition for consistency of the branes is that their self-overlap, and overlaps with other well-defined branes give rise to unitary spectra.¹² We shall focus on the continuous branes since we saw in the last section that they do not have a Ramond-Ramond tadpole at infinity.

We choose a parameterization $J = \frac{1}{2} - iP$, where P is allowed to take complex values. The overlap between two branes can be found by expanding the two branes using the one-point functions (2.4) in the defining Ishibashi basis (2.2). One finds [17], after an exchange of order of integration to which we shall return shortly:

$$\begin{aligned} e^{\pi \frac{3z^2}{T}} \langle B; J_1, M_1 | e^{-\pi T H^{\text{cl}}} e^{i\pi z(J+\tilde{J})} | B; J_2, M_2 \rangle &= \int_{-\infty}^{\infty} dp [\rho_1(p|J_1, J_2) Ch(p, M_2 - M_1; it, z') \\ &\quad + \rho_2(p|J_1, J_2) Ch(p, M_2 - M_1 + 1; it, z')] \end{aligned} \tag{4.1}$$

¹⁰One can instead compare the RR sectors of the above brane and the localized brane — the localized brane $|B; Id\rangle$ rotates along with the potential and the relative orientation of the two branes changes.

¹¹We shall impose the Seiberg bound $J \leq \frac{1}{2}$ following [34].

¹²This problem was touched upon in [17].

where the spectral densities ρ_i are given by:

$$\begin{aligned} \rho_1(p|J_1, J_2) &= \int_0^\infty dp' \frac{\cos(4\pi pp')}{\sinh^2(2\pi p')} \sum_{\epsilon_i=\pm 1} \cosh\left(4\pi\left(\frac{1}{2} + i\epsilon_1 P_1 + i\epsilon_2 P_2\right)p'\right) \\ \rho_2(p|J_1, J_2) &= 2 \int_0^\infty dp' \frac{\cos(4\pi pp')}{\sinh^2(2\pi p')} \sum_{\epsilon=\pm 1} \cosh(4\pi(iP_1 + i\epsilon P_2)p'). \end{aligned} \quad (4.2)$$

For $P_i \in \mathbb{R}^+$, these formulas are well-defined. For imaginary values of P_i which we are interested in, corresponding to $0 \leq J \leq \frac{1}{2}$, one has to be more careful. The p' integral in (4.2) may generate additional divergences at $p' = \infty$, which can be eliminated by shifting the contour of p integration in (4.1) *before exchanging the order of the integrals* as explained in [15]. Thereafter, one can freely exchange the integral and shift back the contour, finding additional contributions to the brane spectrum.

As a warmup, let us view how the above analysis accords with the addition relation for the branes. From the addition relation, we would expect a $J = M = 0$ brane to have the same spectrum as a $J = M = \frac{1}{2}$ brane, with two extra continuous representations at $j = m = \frac{1}{2}$ and another localized mode corresponding to the identity character. This is in accord with the following observation. The densities associated to the $J = 0$ brane and the $J = \frac{1}{2}$ brane are related as follows:

$$\begin{aligned} \rho_1(p|J = 0) &= \rho_1(p|J = \frac{1}{2}) + 4 \int_0^{+\infty} dp' \cos 4\pi pp' \cosh 2\pi p' \\ \rho_2(p|J = 0) &= \rho_2(p|J = \frac{1}{2}) + 4 \int_0^\infty dp' \cos 4\pi pp'. \end{aligned} \quad (4.3)$$

These integrals are divergent if (p, p') are real, but let us imagine having solved this problem by shifting the contour of p in (4.1) for the moment and proceed unhindered. We see that the $J = 0$ self-overlap picks up a delta function contribution at $p = 0$ from the second density ρ_2 , after p -integration, giving rise to one continuous character at $j = \frac{1}{2}$. The first density ρ_1 , after integration over p' , contains two poles, at $ip = \pm \frac{1}{2}$. After shifting (back) the contour of integration of p to the real axis, the integral on the real axis vanishes due to anti-symmetry, and the pole at $ip = \frac{1}{2}$ is picked up in the process, contributing a continuous $j = 0$ character. The latter splits into the second $j = \frac{1}{2}$ character and the identity character. Thus, we see that the annulus spectrum is consistent with the brane addition relation.

We will now put all of this on a firm footing using the ideas of [15]. We first shift the contour of the p integration in the complex plane to make the integral (4.2) well-defined for imaginary p_i as well. The spectral densities ρ_i have divergent pieces independent of the boundary states coming from the region $p' = 0$. These can be subtracted off, and one can define relative spectral densities. We use the regularization of [14] in terms of the q-Gamma function with $b = 1$:

$$\log S_b(x) = \int_0^\infty \frac{dt}{t} \left[\frac{\sinh(Q - 2x)t}{2 \sinh(bt) \sinh(t/b)} - \frac{(Q/2 - x)}{t} \right]; \quad Q = b + \frac{1}{b}. \quad (4.4)$$

The convergent parts of the spectral densities are the following:

$$\begin{aligned} \rho_1(p|J_1, J_2) &= \frac{1}{8\pi i} \sum_{\epsilon_i=\pm 1} \epsilon_0 \partial_p \ln \mathcal{S}_1 \left(\frac{1}{2} + i\epsilon_0 p + i\epsilon_1 P_1 + i\epsilon_2 P_2 \right) \\ \rho_2(p|J_1, J_2) &= \frac{1}{4\pi i} \sum_{\epsilon_i=\pm 1} \partial_p \ln \mathcal{S}_1(1 + ip + i\epsilon_1 P_1 + i\epsilon_2 P_2). \end{aligned} \quad (4.5)$$

Now, on shifting back the contour of integration of p , there is a possibility of picking up additional contributions from the poles of the functions (4.5). For the self overlap, we have $J_1 = J_2 =: J$. Some details of this computation are given in appendix A, and we obtain the following results:

Results of the self overlap computation for the extended branes.

- 1a. For $J > 0$, the function $\rho_2(j)$ defined by (4.2) needs no shift of contour for its convergence. For the case $J = 0$, a pole and a zero in the third and the fourth S_1 functions cancel each other and there is no extra pole.
- 1b. There *is* however a delta function contribution to the function $\rho_2(j)$ at $j = \frac{1}{2} - ip = \frac{1}{2}$. This is due to a crossing of the branch cut of the logarithm in (4.5).
- 2a. For $\frac{1}{4} < J \leq \frac{1}{2}$, the contour in (4.2) for the function $\rho_1(j)$ is well-defined, and there are no extra contributions to the spectrum.
- 2b. For $0 < J \leq \frac{1}{4}$, there is a delta function contribution to $\rho_1(j)$ at $j = \frac{1}{2}$ for the same reason as above.
- 2c. For $J = 0 - \epsilon$, $\epsilon \geq 0$, there is a pole at $j = \frac{1}{2} - ip = 0$ whose residue is unity.

Comments.

- 1. The branes with $J \leq 0$ have in their spectrum new extra localized modes contained in the continuous character $J = M = 0$. Many of these branes (*e.g* $J < 0$) have a non-unitary spectrum in their overlap with the identity brane, as we shall see soon.
- 2. The computation above was for the NS character. The presence of the other three sectors will be dictated by supersymmetry.
- 3. The delta function in ρ_1 and ρ_2 also makes its appearance in the boundary Liouville theory, as is consistent with the addition relation in that context. In fact, the function ρ_2 is exactly the spectral density on the extended branes of Liouville theory with an appropriate change of variables.

4.2 Mutual Supersymmetry and the GSO projection

We have understood the various boundary states in the cigar SCFT from different points of view. Now we want to focus on the properties of the branes in the full six-dimensional string theory. From this point on, we shall discuss these branes, using the construction

of [7, 17] to complete the cigar boundary states into the full D-brane boundary state. We need to make a GSO projection in the open string channel consistent with the closed string spectrum. In [7], we discussed only one type of brane, using the boundary state $|B, Id\rangle$. When there are no other branes present in the background, the two possible GSO projections are equivalent, giving rise to a brane $|D3\rangle$ and an anti-brane $|\overline{D3}\rangle$. We shall denote the full extended branes and anti-branes by $|D5; J, M\rangle$ and $|\overline{D5}; J, M\rangle$.

We are interested in the question of how much supersymmetry is preserved in the presence of one or more of the branes we have described. The closed string background has $\mathcal{N} = 2$ Poincare supersymmetry in $d = 4$, and has a $U(1)_R$ symmetry arising from the rotation of the cigar [38]. The localized brane $|D3\rangle$ and the anti brane $|\overline{D3}\rangle$ preserve half of the eight bulk supercharges. They do not preserve any of the same supercharges and a configuration of a $|D3\rangle$ and a $|\overline{D3}\rangle$ is non-supersymmetric and has a tachyon in the spectrum.

The arguments for supersymmetry in [7] (appendix A) relied basically on the Neumann boundary condition for the R -current of the $\mathcal{N} = 2$ theory. All the branes in question here are B -branes and preserve the Neumann boundary condition, and so are half BPS by themselves. They all conserve the $U(1)_R$ symmetry.¹³ The question of mutual supersymmetry thus boils down to a GSO projection, which can be seen in the (non)vanishing of the annulus diagram between the various branes.

Let us first investigate the supersymmetry of the branes relative to the $|D3\rangle$ brane. We note that the twisted NS and the R-sectors characters follow by spectral flow from the NS sector. After tensoring the flat space parts, we have the following two sets of branes based on the continuous characters with vanishing overlap with the D3-brane (with $h = -J(J - 1) + M^2$ arbitrary):

$$\begin{aligned} \langle D3|e^{-TH_{cl}}|D5; J, M = 0\rangle = \\ \frac{1}{2} \frac{q^{h-\frac{1}{4}}}{\eta^6(\tau)} \left[\vartheta_{00}(2\tau, 2z) (\vartheta_{00}^2(\tau, z) - \vartheta_{01}^2(\tau, z)) - \vartheta_{10}(2\tau, 2z) \vartheta_{10}^2(\tau, z) \right] = 0 \end{aligned} \quad (4.6)$$

$$\begin{aligned} \langle D3|e^{-TH_{cl}}|\overline{D5}; J, M = \frac{1}{2}\rangle = \\ \frac{1}{2} \frac{q^{h-\frac{1}{2}}}{\eta^6(\tau)} \left[\vartheta_{10}(2\tau, 2z) (\vartheta_{00}^2(\tau, z) + \vartheta_{01}^2(\tau, z)) - \vartheta_{00}(2\tau, 2z) \vartheta_{10}^2(\tau, z) \right] = 0 \end{aligned} \quad (4.7)$$

We have mutually supersymmetric branes for any value of the parameter J . Note that the second brane is an anti-brane. This nomenclature is based on the semiclassical notion of the flux measured in the weak coupling region. The two branes above have an opposite sign for the flux (as we will explain in more detail later on). On the open string side though, the GSO projection is the same — so that the two branes are mutually supersymmetric. We present a low energy expansion of these partition sums in appendix B.

Remarks.

¹³The boundary states are invariant under this symmetry, the backreaction onto the background causes this symmetry to be anomalous on the branes [7].

1. All the modes in these expansions are localized because the $|D3\rangle$ brane is.
2. For $J < 0$, both the amplitudes $\langle D3|e^{-TH_{cl}}|D5; J, M = 0\rangle$ and $\langle D3|e^{-TH_{cl}}|\overline{D5; J, M = \frac{1}{2}}\rangle$ have a non-unitary spectrum.
3. For $0 \leq J < \frac{1}{2}$, the amplitude $\langle D3|e^{-TH_{cl}}|\overline{D5; J, M = \frac{1}{2}}\rangle$ has a *tachyonic spectrum*. The degeneracy between bosons and fermions then implies that the fermionic spectrum is non-unitary.
4. For $0 < J < \frac{1}{2}$, the amplitude $\langle D3|e^{-TH_{cl}}|D5; J, M = 0\rangle$ has massive modes.

We are then left with two extended branes with massless modes in their overlap with $|D3\rangle$, the analysis of which we turn to next.

4.3 Summary of branes which realize interesting gauge theories

We have the following list of interesting branes which preserve the same $\mathcal{N} = 1$ supersymmetry in $d = 4$ — the brane $|D3\rangle$, the brane $|\overline{D5; J = M = \frac{1}{2}}\rangle$ and $|D5; J = M = 0\rangle$. The various partition functions contain a four dimensional space filling piece. The massless spectrum in four dimensions is then controlled by the piece of the partition function arising from the cigar. In this subsection, we present some details of the spectra among these various branes and summarize them at the end.

The brane $|D3\rangle$ has only localized modes in its spectrum. The cigar piece of the partition function contains only the identity character Ch_{Id} and the massless fields are a gauge field multiplet in four dimensions [7].

The brane $|\overline{D5; J = M = \frac{1}{2}}\rangle$ has no localized modes in its self-overlap. Its overlap with the $|D3\rangle$ contains from the cigar piece $Ch_{cont}(J = M = \frac{1}{2})$ whose massless spectrum consists of a quark and an anti-quark multiplet [6].

The states $|D5; J, M = 0\rangle$ with $\frac{1}{4} < J < \frac{1}{2}$ have a self-overlap which is exactly the same as that of the $|\overline{D5; J = M = \frac{1}{2}}\rangle$. For $0 < J \leq \frac{1}{4}$, there is in addition one other mode at the boundary of the continuous representation $j = \frac{1}{2}$. In the overlap with the $|D3\rangle$, the cigar part of the character is $Ch_{cont}(J, M = 0)$, which has generically only massive four-dimensional modes for $J > 0$.

As $J \rightarrow 0$, the modes in the overlap with the $|D3\rangle$ start to become massless. The brane corresponding to the boundary state $|\overline{D5; J = M = 0}\rangle$ has in its overlap with $|D3\rangle$ the continuous character $Ch_{cont}(J = M = 0)$ which is a sum of $Ch_{cont}(J = M = \frac{1}{2})$ and Ch_{Id} . The massless fields in this overlap are a quark and anti-quark multiplet from the $J = \frac{1}{2}$ character as well as a gauge field multiplet from the Ch_{Id} . Its self overlap however, is a sum of the self overlap of $|\overline{D5; J = M = \frac{1}{2}}\rangle$, a piece from the character at the edge of the continuum $Ch(j = m = \frac{1}{2})$ and another from the character $Ch(j = m = 0)$. Together, we have in the spectrum a four dimensional gauge superfield (from the lowest mode), and two massless scalars which we interpret below.

A note on Higgsing and bound states. Before we finish this section with a recap of the highlights in a table, we make a few comments on the realization of the Higgs mechanism in this theory and the two massless scalars with $j = m = \frac{1}{2}$ appearing in the self overlap

Brane $ B\rangle$	RR Charge	Overlap with $ D3\rangle$	Self-overlap
$ D3\rangle$	+1	Gauge field A_μ	A_μ
$ D5; J = M = \frac{1}{2}\rangle$	$-\frac{1}{2}$	Quarks Q, \tilde{Q}	No localized modes
$ D5; J = M = 0\rangle$	$+\frac{1}{2}$	A_μ , Quarks Q, \tilde{Q}	A_μ , Meson M , marg. massless

Table 1: Summary of mutually supersymmetric branes with unitary overlaps.

of the $|D5; J = M = 0\rangle$. One of them appears in the self-overlap of the extended brane starting at $J = \frac{1}{4}$ and remains in the spectrum when we decrease $J \rightarrow 0$, the other appears strictly at $J = 0$.

This phenomenon also happens in Liouville theory.¹⁴ In Liouville theory, the full knowledge of the open string correlators showed us that this mode appearing at the edge of the continuum is not a genuine localized mode. We can call it a marginally localized mode. The other mode however is a genuine localized mode and its reflected non-normalizable part actually vanishes [34].

In our case as well, we should think of one of the scalar modes as a marginally massless mode and the other as a genuine one. The latter mode M^{ij} transforms in the adjoint of the flavor group and is colorless, i.e. it has the quantum numbers of a meson. Recall that the closed string theory and our branes conserve in perturbation theory the $U(1)_R$ symmetry corresponding to the rotation of the cigar. The meson has charge $P_\theta = 1$ under this symmetry. This is twice the charge that the quarks carry.¹⁵ We shall understand this better in geometric terms in section 8.

We shall also see in section 8 that the backreaction onto the Ramond-Ramond axion which effectively counts the objects charged under P_θ in the massless spectrum is consistent with the existence of one massless scalar (not two) on the brane $J = M = 0$. This convinces us that the other marginally massless mode does not affect the four dimensional physics.

To summarize, the $|D5; J = 0^+, M = 0\rangle$ brane can be understood as being at the origin of the Higgs branch of the gauge theory. Thought of as a sum of the $|D5; J = M = \frac{1}{2}\rangle$ and $|D0\rangle$, there are massless quarks in the spectrum. There is a parameter on the $|D5; J = M = \frac{1}{2}\rangle$ which gives mass to the quarks. On the other hand, the above way of thinking of this brane as a single object shows that on this branch, the axial combination of the two $U(1)$'s rotating the two branes independently is broken, there is a corresponding massless Nambu-Goldstone boson which can be given an expectation value. This meson has the same quantum numbers as the operator $Q\tilde{Q}$. There are other branes with unitary branes with $0 < J < \frac{1}{2}$ whose details are in the above subsection. In the last two columns, we show only the massless modes. The marginally massless modes are the ones which become massive when the meson expectation value is non-zero.

¹⁴For the same technical reason of the logarithmic branch cut. As we stress later, it can be thought of as capturing the topological part of the full theory we are interested in.

¹⁵Although the quarks and the mesons appear in the same open string character, the quarks have one of their ends on the localized $|D3\rangle$ brane, while the mesons have both their ends on the extended brane. One can think of the difference in charges as being absorbed by a open string vertex operator which implements the change in boundary conditions.

5. Relation to brane set-ups in ten dimensions

We can use our understanding of the closed string parameters to link the non-critical superstring picture to the more familiar brane set-up in ten dimensions. A conventional configuration of branes for the study of the physics of $\mathcal{N} = 1$ gauge theories is the following (see e.g. [39] for a review):

$$\begin{aligned}
 \text{Spacetime} &: \quad 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9. \\
 \text{NS5} &: \quad 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ - \ - \ - \ -. \\
 \text{NS5}' &: \quad 0 \ 1 \ 2 \ 3 \ - \ - \ - \ - \ 8 \ 9. \\
 \text{D4} &: \quad 0 \ 1 \ 2 \ 3 \ - \ - \ 6 \ - \ - \ -. \\
 \text{D6} &: \quad 0 \ 1 \ 2 \ 3 \ - \ - \ - \ 7 \ 8 \ 9.
 \end{aligned} \tag{5.1}$$

Let us first consider a bulk theory without D-branes. The relative motion of the NS5-branes is possible in the non-compact directions (x^6, x^7) . The motion in these directions is captured in the non-critical string theory in six dimensions by the parameters $\mu, \bar{\mu}$ of the $\mathcal{N} = 2$ Liouville theory. The string coupling is set by the absolute value of μ which measures the distance between the NS5-branes, while the orientation of their relative position in the (x^6, x^7) plane is set by the phase of μ . The exchange of the NS5-branes is implemented by the map $\mu \rightarrow -\mu$.

We can gather further evidence for this identification after introducing the D4-brane into the set-up. There are three gauge theory parameters for pure Yang-Mills theory — the gauge coupling and theta angle $\frac{1}{g_{\text{YM}}^2} + i\theta$, and the Fayet-Iliopoulos parameter r . These three quantities can be thought of as the values that closed string fields take on the brane. In the non-critical superstring theory, the relevant string modes are the complex tachyon T, \bar{T} and the RR axion χ . For $\mu = \bar{\mu}$, the modes $(T + \bar{T}, \partial_+ \chi + \partial_- \chi)$ fall into a $\mathcal{N} = 1$ multiplet of the preserved supersymmetries [38]. The conserved supersymmetries transform covariantly under a rotation of the cigar, and so do the corresponding combinations of the fields.

The non-critical superstring theory is obtained from the ten-dimensional string in a double scaling limit (after a T-duality), in which the mass of the D4-brane stretching between the NS5-branes is kept fixed while scaling down the length of the D4-branes, and the string coupling simultaneously [10]. The fixed mass of the D4-brane sets the parameter μ , which also sets the gauge coupling. (The phase of μ does not enter the physics. Also, with no flavours present, the FI parameter cannot be turned on while preserving supersymmetry. We recall that we have seen in [7] that the θ angle is associated with the zero modes of the RR axion field χ .)

When we further introduce D6-branes, the relative orientation of the D6-branes and the NS5 branes in the $(6, 7)$ plane becomes important. Equivalently, the relative orientation of the D6-branes and the D4-branes stretching between the NS5-branes is fixed by the requirement of supersymmetry.

In the non-critical superstring set-up, we certainly have a supersymmetric configuration when the phase of the D5 brane boundary state agrees with that of the D3 brane boundary state (in the sense that the overlap is the supersymmetric one given in the previ-

ous section). This corresponds to the D6-branes being orthogonal to the D4-branes in the ten-dimensional picture. We can keep the D6-branes fixed while rotating the NS5-branes (and the D4-brane in between), by changing the phase of μ and by shifting the angular brane parameter M simultaneously, thus breaking supersymmetry.

We can thus identify (while keeping in mind that we agree to rotate the parameter M of the D6-branes along with the phase of μ) that the parameter $\frac{1}{2}(\mu + \bar{\mu})$ can be associated to the gauge coupling on the D4-brane, which is the motion in x^6 (or rather, the length of the localized D4 branes in the x^6 direction), while the motion in x^7 is the FI parameter which we associate with $\frac{1}{2i}(\mu - \bar{\mu})$.

5.1 The branes in the non-critical string theory

Before re-interpreting our non-critical brane set-up in ten dimensions, let's turn to the brane addition relations in the six-dimensional non-critical superstring. We first remark that on subtracting the equations (4.6) , (4.7) with $J = M = 0$ and $J = M = \frac{1}{2}$, we recover the partition function of the D3-brane:

$$\langle D3|e^{-TH_{cl}}|D5; J = M = 0\rangle - \overline{\langle D3|e^{-TH_{cl}}|D5; J = M = \frac{1}{2}\rangle} = \langle D3|e^{-TH_{cl}}|D3\rangle. \quad (5.2)$$

We can also use the addition relation of our previous section tensored with the same space filling brane in flat space to get the brane addition relation in the six-dimensional non-critical superstring theory:

$$|D5; J = M = 0\rangle = \overline{|D5; J = M = \frac{1}{2}\rangle} + |D3\rangle \quad (5.3)$$

We will comment later on the addition relation with the branes of opposite charges.

Moreover, we recall that the first order backreaction onto the cigar was calculated in [7] , and it was found that the $|D3\rangle$ sources the RR axion. We will see later in section 8 that the $|D5, J = M = 0\rangle$ and $\overline{|D5, J = M = \frac{1}{2}\rangle}$ also source the RR axion with charges, in units of the $|D3\rangle$ brane charge $+\frac{1}{2}$ for the first and $-\frac{1}{2}$ for the second brane.

5.2 The identification

We are ready to identify the branes in the non-critical superstring theory with the branes in the original (not yet doubly scaled, and T-dual) ten-dimensional superstring set-up. For definiteness, let's say the NS5 is to the left of NS5' in the x^6 direction. Then we have the following identifications:

1. The $|D3\rangle$ maps onto [6] a D4 brane starting on NS5 and ending on NS5'. We assign to it a charge +1. (This fixes the orientation of the brane.)
2. The brane $\overline{|D5, J = M = \frac{1}{2}\rangle}$ maps to [6] a D6 brane to the left of the NS5 with a D4 brane starting on it and ending on NS5. This has charge $-\frac{1}{2}$.
3. $|D5, J = M = 0\rangle$ is a D6 brane to the left of the NS5 with a D4 brane starting on it and ending on NS5'.

The first identification is clear. The brane is the only localized D-brane consistent with unitarity and supersymmetry which carries the right spectrum, namely a $\mathcal{N} = 1$ vector multiplet. For the second and third identifications, the arguments are the following. Firstly, these are extended branes, reaching asymptotic infinity. In the picture T-dual to the cigar, they reach asymptotic infinity as a particular line in the (x^6, x^7) plane. This agrees with the asymptotic form of the D6-branes in this plane. Secondly, we fixed the charge of the D4-brane before, and the charges listed above are then found by simple computation. These charges agree with those of the D6 branes. The third argument is that the spectra on these branes agree precisely in the non-critical and in the ten-dimensional picture. Moreover, a fourth argument is that indeed, these branes are consistent with the first identification we made, in that these three classes of branes do satisfy the addition relation $|D5, J = M = 0\rangle = |\overline{D5}, J = M = \frac{1}{2}\rangle + |D3\rangle$. The charges add appropriately $-\frac{1}{2} + 1 = \frac{1}{2}$.

Thus, we are now equipped to study the movements of the brane in the ten-dimensional brane set-up, to translate these into the non-critical superstring theory, and to analyze the effect on the gauge theories living on the branes in terms of their exact boundary state description.

6. Electric-Magnetic duality in the gauge theory

In this section we discuss electric-magnetic duality in $\mathcal{N} = 1$ supersymmetric quantum chromodynamics, within the framework of non-critical superstring theory. We have seen how the $|\overline{D5}; J = M = \frac{1}{2}\rangle$ brane introduces supersymmetric quarks to the gauge theory on the $|D3\rangle$ brane. We assumed that μ is positive in our identification of these boundary states as positive tension branes. We also saw that for $\mu > 0$, the brane $|D5; J = M = 0\rangle$ introduces quarks as well as mesons to the gauge theory on the $|D3\rangle$ brane.

Although we will not further need the identification of these branes in ten-dimensions, it may be useful for the reader to keep in mind that we identified these boundary states in the ten-dimensional set-up, under the assumption that the NS5-brane is to the left of the NS5' brane. Namely, we identified these boundary states as corresponding to certain branes in the electric picture. The fact that the NS5-brane is to the left of the NS5' brane is equivalent to restricting to positive values of μ . We want to study now what happens to the configuration of branes as we go to negative values of μ (purely within the exact description of the branes in the non-critical superstring theory).

We start with an electric configuration of N_f electric flavor branes and $N_c < N_f$ color branes, and perform the operation $\mu \rightarrow -\mu$ on the system. Using the mapping of boundary states described in equation (3.3), we obtain:

$$\begin{aligned}
 |D3; \mu\rangle &\rightarrow -|D3; -\mu\rangle; \\
 \overline{|D5; J = M = \frac{1}{2}; \mu\rangle} &\rightarrow |D5; J = M = \frac{1}{2}; -\mu\rangle \\
 &= |\overline{D5}; J = M = 0; -\mu\rangle - |\overline{D3}; -\mu\rangle.
 \end{aligned}
 \tag{6.1}$$

Now, for negative μ , which we associated to the NS5-branes being to the right of the NS5' branes in the x^6 direction, we will identify these boundary states differently with branes in ten dimensions. In particular, we will now refer to these branes as being magnetic branes.

The final configuration is then N_f magnetic branes $|\overline{D5}; J = M = 0; -\mu\rangle$, N_f color anti-branes $(-\overline{D3}; -\mu)$ and N_c color branes $(-D3; -\mu)$ (Remember that both of these localized branes are well-defined positive-tension objects). Considering that the branes and anti-branes annihilate each other and that the left-over purely closed string configuration decouples from the gauge theory physics, we find [40]:

$$|\overline{D3}\rangle + |D3\rangle \rightarrow \text{closed string vacuum} + \text{closed string decay products}, \quad (6.2)$$

and we are left with a mutually supersymmetric system of N_f magnetic anti-branes $|\overline{D5}; J = M = 0\rangle$, and $N_f - N_c$ color anti-branes $-\overline{D3}$. Thus, the gauge theory of the dual configuration of branes at negative values of μ is the Seiberg dual. Note how the various reversals of sign in (3.3) naturally lead to a reversal of the addition relation in the dual configuration, and a set of final states consistent with charge conservation.

Note that for $N_f < N_c$, we cannot condense the open string tachyon fully by this process. In this case, we break supersymmetry. In fact, we could describe the above movement of branes as a function of μ in the complex plane. For $N_f > N_c$, the movement can be performed while preserving supersymmetry along the whole path. While changing the phase of μ , we would rotate the parameter M of the extended branes as well, keeping the extended branes fixed at infinity. As we rotate, we need to recombine the extended branes with the localized branes, turning on a vev for the strings stretching between N_c of the localized and the extended branes. It should be possible to show in detail that these vevs can be turned on consistent with supersymmetry only when $N_f > N_c$. This follows from the effective action, but it is feasible as well to show this explicitly using the full boundary state. When $N_c < N_f$, we will not find a sufficient number of these open string modes to preserve supersymmetry while rotating μ .

6.1 A note on the deformations of the theory

In the electric and magnetic versions of the gauge theory, it is well-known that the deformations in one theory map to expectation values in the other. In the simplest case, the mass parameter for the quarks in the electric theory has the same quantum numbers as the expectation values of the meson in the magnetic dual. This relation is realized in the open string theory on the branes in a rather interesting manner.

In the electric description of the theory, the parameter J controls the mass of the quarks for $M = \frac{1}{2}$. When $J \rightarrow \frac{1}{2}$, the quarks become massless and can condense. The mass deformation is described by the zero mode of a field moving on the extended brane. The vertex operator for this field is described by the non-normalizable mode corresponding to $h = Q = 0$ tensored with an excitation in the cigar directions from the $(j = m = \frac{1}{2})$ character. In the dual description, the vertex operator for the open string field corresponding to the dual localized meson is almost the same as above with the only difference that the $h = Q = 0$ mode on the cigar takes on the normalizable branch.

7. Tachyon condensation and boundary RG flow

We have understood the brane addition relation for a closed string theory μ

$$|D5; J = M = 0; \mu\rangle = \overline{|D5; J = M = \frac{1}{2}; \mu\rangle} + |D3; \mu\rangle, \quad (7.1)$$

where all the objects in the above equations have positive tension. We would like to analyze further the analogous relation for mutually non-supersymmetric branes in a given closed string theory at a fixed value of the parameter μ . The first of the above equations (7.1) implies

$$\begin{aligned} |D5; J = M = 0\rangle + |\overline{D3}\rangle &= \overline{|D5; J = M = \frac{1}{2}\rangle} + |D3\rangle + |\overline{D3}\rangle. \\ &\longrightarrow \overline{|D5; J = M = \frac{1}{2}\rangle}. \end{aligned} \quad (7.2)$$

where, in the first line, the right hand side manifestly does not preserve any supersymmetry of the bulk theory. It contains a non-supersymmetric brane without RR charge (which can decay to the closed string vacuum).

The general arguments of decay of a brane and its anti-brane to the closed string vacuum are given in the context of open string field theory, and it is understood explicitly as a time-dependent process [40]. We wish to argue now that we can actually understand the above equation from the left hand side directly to the second line as a boundary renormalization group flow on the $J = M = 0$ brane. We first argue this by relating the set-up to a similar configuration in bosonic Liouville theory, where the worldsheet boundary RG flow has been well-understood. Then, we discuss how this could be related to a boundary RG flow within the cigar boundary conformal field theories.

Firstly, let's discuss how to link up the relation between non-supersymmetric boundary states with a boundary renormalization group flow in bosonic Liouville theory. By appropriately twisting the $\mathcal{N} = 2$ Liouville worldsheet theory, we can focus on the topological subsector of our closed string background which is described by a bosonic string theory with a $c = 1$ boson at self dual radius coupled to Liouville theory [41, 42]. In [16], it was shown how to extend this map to the open string sector, mapping the boundary states and the boundary two-point functions.

The topological subsector of the BPS branes we are studying is thus described by the branes in Liouville theory. The $|D3\rangle$ branes map to the ZZ branes and the extended branes $\overline{|D5; J = M = \frac{1}{2}\rangle}$ and $|D5; J = M = 0\rangle$ map to the FZZT branes labelled by $\sigma = 1$ and $\sigma = 0$ of [34] respectively. The addition relation we describe above, restricted to the topological subsector simply maps to the addition relation of [22, 34] under the twist. The spectrum on the $\sigma = 0$ brane contained in addition to the continuous modes, a localized mode. In the physical non-critical superstring theory, this mode lifts to an infinite set of open string modes summarized by the character $Ch(J = M = 0)$ which gives rise to the meson multiplet.

We can now try to understand the open string tachyon condensation describing the process in (7.2) in this subsector. The paper [34] described how to understand the loss of

the localized brane as a boundary RG flow. The RG flow is seeded by the highly relevant (dimension zero) operator which was the new mode generated on the $\sigma = 0$ brane. In our case, the operator to which this is lifted is not present in the spectrum of the BPS branes, due to the GSO projection. However, the spectrum of open strings between the non-mutually supersymmetric branes $|D5; J = M = 0\rangle$ and $|\overline{D3}\rangle$, has the opposite GSO projection and does contain the relevant boundary perturbation as its lowest mode, the tachyonic dimension zero operator B_0 . In the topological subsector of the Yang-Mills theory we therefore understand the non-supersymmetric brane addition relation (7.2). In terms of the pictures in the previous section, we now have the brane folding back on itself and annihilating a little piece of itself.

To carry this over to the full theory, we would need to understand the structure of the three point functions in the boundary $\mathcal{N} = 2$ Liouville theory, and the renormalization group flows between different boundary conformal field theories. However, we can already abstract lessons from the bosonic Liouville theory example and list the properties which will ensure that the RG flow seeded by the above tachyon proceeds according to our expectations:

1. The bulk-boundary correlators for the extended branes are analytic in the boundary parameters J_i .
2. The dimension zero boundary operator B_0 which is localized on the $J = 0$ brane will act as a projection operator in the boundary Hilbert space.
3. The boundary operator B_0 , when inserted in correlators involving the boundary state $J = 0$ will act as a projector onto the localized brane.
4. There is an exact boundary renormalization group flow from the boundary state $J = 0$ to the $J = 1/2$ brane, under perturbation by B_0 .
5. Thus, the boundary RG flow removes the part of the boundary state $J = 0$ that is picked up under monodromy, under perturbation by the localized mode.

These properties form an important ingredient in a microscopic understanding of Seiberg duality and it would be interesting to demonstrate them beyond our analysis in the topologically twisted sector (using the results of [34]).

8. What *is* the theory on the branes? — Global symmetries and RG flows

So far, we have argued that the gauge theories realized as low energy limits of the two brane configurations belong to the same moduli space. To really argue for a full IR equivalence as in [11] between the two descriptions, one must show that any open string process that contributes in the extreme IR is independent of $sign(\mu)$.¹⁶

¹⁶This is certainly true for some simple open string processes like those involving only gauge fields and its superpartners, and for the mass terms and the quartic coupling of the quarks.

In this section, we try to understand more precisely the relation between the open string theory living on the worldvolume of the D-branes and the electric/magnetic descriptions of $\mathcal{N} = 1$ SQCD. To this end, we first present a list of the global symmetries and charge assignments of the various fields. We then compute the backreaction onto the closed string background. We match the backreaction onto the RR axion with an anomaly coefficient in the field theory which depends only on the massless spectrum. Then we use the backreaction onto the NSNS background to teach us about the interactions in the theory.

From the analysis of the open string theories, we know that the two low energy theories under consideration have the same field content as the electric and magnetic descriptions of SQCD. However in our construction, we do not have another parameter to tune the QCD scale relative to the string scale and it is determined dynamically. We can think of the gauge theory on the branes as being completed by an open string field theory. More practically, this theory can be defined with a string scale cutoff with natural values at that scale for all allowed interactions.

Although there are only a few terms in the action (corresponding to the “pure” SQCD) which are dimensionally relevant, the running to strong gauge coupling invalidates this analysis based on perturbation theory. In fact we know [43] that the quartic operator of quark superfields $W_{\text{quart}} \sim Q\tilde{Q}Q\tilde{Q}$ although classically irrelevant actually could gain a large anomalous dimension. Quantum mechanically, this operator depends crucially on the parameters N_f and N_c . SQCD with this quartic coupling of the quarks flows to pure electric SQCD if $N_f - 2N_c > 0$, but to pure magnetic SQCD if $N_f - 2N_c < 0$ [44].

We would like to argue that the theory being described is indeed $\mathcal{N} = 1$ SQCD with a quartic coupling¹⁷ [43], and that our analysis leads to an exact statement of duality between two theories described by different values of the coupling.

It is difficult to check these statements directly in the full string theory, since at present we have little knowledge of the three and four point functions of open strings in these backgrounds. However, we can use the open-closed string duality to gain some insight in how the different behaviours of the gauge theory depending on the sign of $N_f - 2N_c$ are coded in the closed string background. The first order backreaction on the cigar background can be calculated as for the case of pure $\mathcal{N} = 1$ super-Yang-Mills in [7]. We merely sketch the calculation here, since it is very analogous to the detailed discussion in [7].

8.1 Backreaction onto closed string background

To measure the backreaction, we use a similar contour prescription for the integral as in [7] and one has to basically evaluate the one-point function multiplied by the profile of the field at a specific value of the momentum p' where the integrand has a pole. Since we have measured already the backreaction of the localized brane in [7], the relevant quantity is the

¹⁷In this context, see also [5] which describes a very closely related setup in supergravity. One would in fact like to argue that our non-critical setup captures the near-singularity region of the setup of [5]. We thank C. Nunez for a discussion on this issue.

ratio of one-point functions¹⁸ for the extended $|D5, p, Q\rangle$ (2.4) and the localized $|D3\rangle$ (2.3):

$$\begin{aligned} \frac{\psi_{p,Q}(p', Q' = 0)}{\psi_{\text{Id}}(p', Q' = 0)} &= \frac{\cos 4\pi p p'}{\sinh 2\pi p' \tanh \pi p'} \\ \frac{\psi_{p,Q}(p', Q' = 1)}{\psi_{\text{Id}}(p', Q' = 1)} &= \frac{\cos 4\pi p p' e^{-i\pi Q Q'}}{\sinh 2\pi p' \coth \pi p'} \end{aligned} \tag{8.1}$$

To measure the backreaction for the NSNS tachyon with winding number one, and the constant mode of the RR field strength, we need to evaluate the above expressions at $(ip' = 0, Q' = 1)$, and to measure the backreaction onto the metric and dilaton, we need to evaluate them at $(ip' = \frac{1}{2}, Q' = 0)$. Plugging in these values, we find that all three of the above measurements — the flux of the RR one-form field strength, the backreaction onto the graviton-dilaton and the backreaction onto the winding tachyon have the following measurements in the units of the D3 brane flux¹⁹:

$$\begin{aligned} |D3\rangle : & +1 \\ |D5, J = M = \frac{1}{2}\rangle : & -\frac{1}{2} \\ |D5, J = M = 0\rangle : & +\frac{1}{2} \end{aligned} \tag{8.2}$$

For N_c color branes and N_f flavor electric branes, we find that that the backreaction onto the tachyon winding mode, the dilaton and the RR axion flux are all proportional to $(2N_c - N_f)$. Note that the evaluation of the backreaction on the winding number one tachyon and the RR scalar is not sensitive to the precise value of the brane parameter p — it only depends on the quantized brane parameter Q (in contrast to the backreaction on the metric and dilaton).

The backreaction onto the RR axion implies a non-zero theta angle for the gauge theory²⁰ $\theta_{\text{cig}} = 2\theta_{\text{YM}}$ through the coupling of D-instantons [7]. One can also understand in geometric terms how the rotation of the cigar affects the various open string modes. Under such a rotation, the quark which lives on a single sheet of the D2-brane goes around once on a rotation by 2π at infinity while the gluino and meson which live on a double sheeted cover of the circle go around twice under the same rotation.

In the gauge theory, the anomaly in the conservation of the current which rotates the gluini by one unit and the quarks and the anti-quarks in the same direction by half a unit is proportional to $(2N_c - N_f)$ [see e.g. [45]]. It is nice check therefore on the consistency of the whole set-up, that the backreaction onto the zero mode of the RR axion $\chi = (2N_c - N_f)\theta$

¹⁸We write here the ratio of the NS-NS sector wavefunctions, there are related expressions for the RR sector.

¹⁹Of course, finding the values for the first two branes is enough to find the third because of the addition relation (2.7).

²⁰As a reminder, this factor of two arises because of the change of variables between the asymptotic variables and the smooth tip. A rotation of 2π at infinity corresponds to a rotation by 4π near the tip, and the two angles measured respectively phases in the closed string theory and open string theory on the localized brane.

indeed measures the anomaly in the conservation of the naive R current which rotates the cigar direction θ [7].

We can also check now that these charge assignments indeed generate anomaly coefficients for the rotation $U(1)_R$ consistent with (8.2) in the gauge theories on the corresponding branes. In particular, for the $|D5; J = M = 0\rangle$, the potential presence of another localized mode with non-zero charge under P_θ would be inconsistent with the above analysis. This ties up the loose end of the argument in section 4 for the absence of the marginally massless mode in the low energy physics.

Comments.

1. Measuring the backreaction onto the axion in the corresponding dual magnetic picture ($\tilde{N}_c = N_f - N_c, \tilde{N}_f = N_f$) gives the same answer (considering that the charges of all the branes have changed sign).
2. The backreaction can really be trusted far from the tip of the cigar, which corresponds to the UV of the open string theory. We might think of the backreaction of one of the NSNS modes as measuring the running of the gauge coupling $1/g_{\text{YM}}^2$. In the region where we can trust this computation, we deduce that the theory on the D-brane at the corresponding energy scale *is not* pure electric SQCD (which has a first order beta function proportional to $3N_c - N_f$).
3. For $N_f = 2N_c$, we have an extra conserved $U(1)$.

8.2 Global Symmetries and charges

For N_f flavor branes there is a global symmetry $SU(N_f)$ rotating the brane basis. For generic flavor branes at $J = \frac{1}{2} + iP$, it is clear from our semiclassical discussion (see section 3) of the branes folding over and ending before the tip that one cannot rotate the two sheets of the brane independently. This is consistent with the low-energy theory having a mass term to the quarks. As $P \rightarrow 0$, the self-overlap (4.5) changes smoothly and there is no extra massless mode that appears at this point. As we move onto the branch $0 \leq J \leq \frac{1}{2}$, there are extra modes appearing at $J = \frac{1}{4}$ and $J = 0$, but these as we saw are localized and have to do with effects in the four-dimensional theory, the bulk densities ρ_i actually do not change at all. We view this as an indicator that there is no symmetry enhancement happening in the six-dimensional theory.²¹

There are other $U(1)$ global symmetries of SQCD which are usually written as $U(1)_B \times U(1)_a \times U(1)_x$ [9]. The charges under (B, a, x) of the quarks, gluini and meson are $(\pm 1, 1, 0)$, $(0, 1, 1)$ and $(0, 0, 2)$ respectively. The first one, baryon number is exactly conserved. The latter two which are both R symmetries are not exact symmetries, only a linear combination $R = (1 - \frac{N_c}{N_f})a + (\frac{N_c}{N_f})x$ is [11]. In the brane picture, this is understood as the fact that although the rotation of the cigar is broken by the presence of the branes, there *is* a conserved charge involving a rotation of the brane itself with respect to the background [6].

²¹In particular, we never see a $SU(N_f) \times SU(N_f)$ symmetry.

The charges in the brane setup we consider are the geometric rotation P_θ , and the rotation of the left and right part of the extended brane²² which we combine into their sum and difference Q^\pm . The charge Q^+ which rotates the quarks in opposite ways is the baryon number B . Based on our analysis of the theta angle above and the geometric argument, we understand that the rotation of the cigar is generated by the charge $P_\theta = \frac{1}{2}(a+x)$. It is also easy to deduce $Q^- = \frac{1}{2}(x-a)$. The charges under (B, P_θ, Q^-) of the quarks, gluini and meson are $(\pm 1, \frac{1}{2}, -\frac{1}{2})$, $(0, 1, 0)$ and $(0, 1, 1)$ which indeed lead to the usual (B, a, x) assignment.

With these charge assignments, the superpotential of the type $W \sim MQ\tilde{Q}$ is allowed (has charge two) in the magnetic theory, both by the classical R -charge P_θ of rotation of the cigar, and by the exact R -charge of [11]. The rotation P_θ has a natural interpretation as an R -symmetry in the closed string background, since the fermions are naturally anti-periodic around the smooth cigar. The above coupling of quarks and mesons has total charge $Q^- = 0$, but unlike P_θ there is no natural reason to expect Q^- to be an R -symmetry even in the classical theory in our six-dimensional setup.

Our background which has a tachyon winding mode condensate has also a closed string excitation X of the tachyon with $P_\theta = 1$. This would have a mass of string scale and naturally couple to the quarks as $W \sim X^2 + \tilde{Q}XQ$. This superpotential is not forbidden by the classical R -symmetry P_θ , and is a further piece of evidence for the existence of the quartic superpotential (at low energies, after integrating out the massive field).

8.3 Softly broken $\mathcal{N} = 2$ theories?

The $\mathcal{N} = 1$ SQCD with a quartic coupling can be thought of as arising from a low energy description of the softly broken $\mathcal{N} = 2$ gauge theory with N_c colors and N_f matter (hyper)multiplets after integrating out the massive adjoint field. This theory indeed has a beta function proportional to the above coefficient $2N_c - N_f$, and has a global flavor symmetry $SU(N_f)$. This gains further support if one thinks of lifting our whole configuration to ten or eleven dimensions. The six dimensional background can be thought of as arising from taking two parallel NS5-branes in ten dimensions (or M5-branes in eleven) and rotating them in two of the dimensions till they are perpendicular [46]. A non-zero angle of rotation corresponds to giving a mass to the matter in the vector multiplet (softly breaking the theory) and rotation by $\pi/2$ implies the mass is of string scale.

9. Final remarks

In this paper, we analyzed in more detail the D-brane boundary states for theories in four dimensions with $\mathcal{N} = 1$ supersymmetry and N_c colors and N_f flavors, and their behavior under bulk and boundary transformations. In particular, the transformation exchanging NS5-branes in the bulk gave rise to a microscopic description of Seiberg-duality. We discussed a brane addition relation, and also argued for the existence of a boundary

²²Although there is no $SU(N_f)^2$ symmetry on the branes, there is, in perturbation theory a $U(1)^2$ as can be seen from the fact that there are open strings carrying half-integer momentum in the partition function. This is a phenomenon related to the double covering of the brane and exists even for the $J = \frac{1}{2} + iP$ branes.

renormalization group flow of the cigar (or $N = 2$ Liouville) conformal field theory that should encode the projection of a sum of branes onto one of its terms.

Furthermore, we showed that the closed string backreaction captures the qualitative difference in the behavior of the gauge theory under RG flow, depending on the sign of the quantity $N_f - 2N_c$ as for theories with a superpotential quartic in the quark superfields. We also presented other evidence for the presence of this operator — the (absence of) chiral flavor symmetry, the absence of any classical $U(1)$ symmetry forbidding such an interaction, and the enhancement of symmetry at $N_f = 2N_c$.

We would like to make a remark about the “s-rule” in these systems. In the ten-dimensional setup, this would say that there can be at most one supersymmetric $D4$ brane connecting the $D6$ and the $NS5$ branes. In our configuration, we never see explicitly a free localized $|D3\rangle$ brane on the corresponding extended brane $|D5; J = M = \frac{1}{2}\rangle$ (though it carries the corresponding $|D3\rangle$ charge). The s-rule is in this sense trivially realized for this object. What is interesting however is that the other extended brane $|D5; J = M = 0\rangle$ which already carries a localized $|D3\rangle$, does not admit²³ any more. This suggests a corresponding s-rule for the $D6 - NS5'$ system. This is another pointer towards an allowed quartic quark coupling. Indeed, there is such an s-rule for the $D6 - NS5'$ system with a rotated $NS5'$ which realizes for a finite mass adjoint scale coupled to the quarks [47].

Finally, some open problems include:

1. The full proof of the $\mathcal{N} = 2$ Liouville boundary RG flow and the associated properties of the boundary conformal field theory.
2. The extension of our analysis to include orientifold planes, and to realize Seiberg duality in Sp and SO gauge theories microscopically.
3. The extension to gauge theories in other dimensions (e.g. three-dimensional gauge theories on a $D2 + D4$ -brane system in type IIA non-critical string theory).
4. The precise identification of the open string marginal operators involved in the rotation of the parameter μ , consistently with supersymmetry.
5. Understanding the closed string background after the full backreaction of the color branes, and the special nature of the theory at $N_f = 2N_c$.

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²³Recall that the spectrum on the branes with $M = \frac{1}{2}$ becomes non-unitary at $J < 0$.

A. Self-overlap of extended branes

We look for poles and the corresponding residues of the spectral density functions defined by (4.5) in the region where we deform the contour. As explained in [15], the contour has to be shifted from $\max(p_1, p_2)$ on the imaginary axis to zero. We look for poles in that range: $0 \leq ip \leq J$ where $0 \leq J \leq \frac{1}{2}$. To count the number of such poles, the following facts will be useful:

1. If a function behaves near x_0 as $f(x) \sim (x - x_0)^a$, then $\partial_x \log f(x) \sim \frac{a}{(x-x_0)}$ near $x = x_0$. So, the net order of the zero of $f(x)$ (zeros — poles) is the residue of the function $\partial_x \log f(x)$.
2. The function $\mathcal{S}_1(x)$ has simple poles at $x = 2 + m + n$ and simple zeros at $x = -m - n$, with $m, n \in \mathbb{Z}_{\geq 0}$.

Putting $p_1 = p_2$ in (4.5), we get

$$\begin{aligned} \rho_1 &= \frac{i}{2\pi} \partial_p \log \frac{\mathcal{S}_1(\frac{1}{2} + ip)^2 \mathcal{S}_1(\frac{1}{2} + i2p_1 + ip) \mathcal{S}_1(\frac{1}{2} - i2p_1 + ip)}{\mathcal{S}_1(\frac{1}{2} - ip)^2 \mathcal{S}_1(\frac{1}{2} + i2p_1 - ip) \mathcal{S}_1(\frac{1}{2} - i2p_1 - ip)}, \\ \rho_2 &= \frac{i}{2\pi} \partial_p \log \mathcal{S}_1(1 + ip)^2 \mathcal{S}_1(1 + i2p_1 + ip) \mathcal{S}_1(1 - i2p_1 + ip). \end{aligned} \tag{A.1}$$

and we look for poles in these functions between $p = p_1$ and $p = 0$, with $0 \leq ip_1 \leq \frac{1}{2}$. (See the bulk of the paper.)

B. Expansion of some brane overlaps

We present here the low energy expansion of the overlaps between the branes $|D5; J, M\rangle$ with $|D3\rangle$. Let us write a generic partition sum as $Z = Z^{\text{NS}} + Z^{\widetilde{\text{NS}}} + Z^R + Z^{\widetilde{R}}$. The low energy expansions of these functions are (with $h = -J(J - 1) + M^2$):

$$\begin{aligned} \mathbf{M} &= \mathbf{0} \\ Z^{\text{NS}} &= q^{h-\frac{1}{2}} + 4q^h y + \dots \\ Z^{\widetilde{\text{NS}}} &= -q^{h-\frac{1}{2}} + 4q^h y + \dots \\ Z^R &= -4q^h y + \dots \\ Z^{\widetilde{R}} &= 0 \end{aligned} \tag{B.1}$$

$$\begin{aligned} \mathbf{M} &= \frac{\mathbf{1}}{\mathbf{2}} \\ Z^{\text{NS}} &= 2q^{h-\frac{1}{2}} y - 8q^h y^2 + 4q^{h+\frac{1}{2}} y^3 + \dots \\ Z^{\widetilde{\text{NS}}} &= 2q^{h-\frac{1}{2}} y + 8q^h y^2 + 4q^{h+\frac{1}{2}} y^3 + \dots \\ Z^R &= -2q^{h-\frac{1}{2}} y + 0q^h y^2 - 4q^{h+\frac{1}{2}} y^3 + \dots \\ Z^{\widetilde{R}} &= 0. \end{aligned} \tag{B.2}$$

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